Computational Aesthetics in Graphics, Visualization and Imaging (2005)
L. Neumann, M. Sbert, B. Gooch, W. Purgathofer (Editors)

Straight away on curved spaces

Alain Marty

Ecole Nationale Superieure d'Architecture Montpellier, France

Abstract

Two points of a Euclidean space can always be connected by the simplest of the curves, the straight line. This curve is unique (it's required), with a constant direction and a minimal length between the two points. The straight line is known to be fundamental in the building of the Euclidean geometry, for the production of all imaginable figures, from the simple triangle to the most complex curved shapes. And when the figures have to be immersed in curved shapes, out of a Euclidean space and without its useful straight line tool, it seems natural to search for a similar fundamental curve possessing its basic properties : uniqueness of the curve, constant direction, minimal length. This curve, known to be a geodesic line, is defined by a differential system whose solutions cannot generally be expressed in a simple way, (let's say as a finite degree polynomial expression), and so leads to a great complexity in the definition of figures belonging to curved spaces. But other approachs exist...

The present contribution, based on a conceptual tool for creating and managing curved shapes, the Pascalian Forms, or pForms, (published in Editions de l'Esperou / Montpellier / 2004), an attempt to generalize the de Casteljau algorithm, focuses on three cases of "straight lines" drawn on curved shapes :

1) the first case shows how a geodesic can be the solution to a very practical problem : applying a long thin plank on a toroidal roof upon a swimming pool ;

2) the second case shows how an apparently complex spatial curve (the Threefoil Knot Curve) followed by a staircase in the MC Escher style can be simplified by using immersed pSegments ;

3) the third case shows how the natural/organic shapes in the Sagrada Familia Temple dreamed by the catalan architect A. Gaudi have been mastered using ruled surfaces (pSurfaces).

Categories and Subject Descriptors (according to ACM CCS): I.3.3 [Computer Graphics]: Curved Shapes Generation

1. The pForms, basics

The approach chosen in Pascalian Forms enlightens a special family of curved shapes, the pForms, built in an affine space (without metric) of any dimensions (actually limited to 4) ; pCurves, pSurfaces, pVolumes are easy to define and manage with a small number of operators, the most fundamental of them simply returning the middle of two points. With this operator used recursively, it is obvious to build segments (Figure 1), hyper-paraboloids (Figure 2), twisted cubes (Figure 3), hypercubes, and so on.

1.1. A hyperbolic paraboloid (pS22) and its diagonal path (pL3)

Some experiences can be done on this first family of recursive multilinear shapes. It is easy to split a hyperbolic paraboloid (called pS22 to be short) in four equals sub pS22, and to do it again recursively along a diagonal path (Figure 4, left). Linked to the 4 controling points (p00,p01,p10,p11) of the pS22, we can consider 3 new points (p0,p1,p2), (Figure 4, right) :

$$p0 = p00,$$
$$p1 = (p01 + p10)/2,$$
$$p2 = p11.$$

build their middle :

$$pM = (p0 + 2 * p1 + p2)/4$$

and do it again recursively, according to the *de Casteljau algorithm*, (Figure 5). Embedded in the pS22, the diagonal path can be considered as a straight line (ipL2/pS22), or as

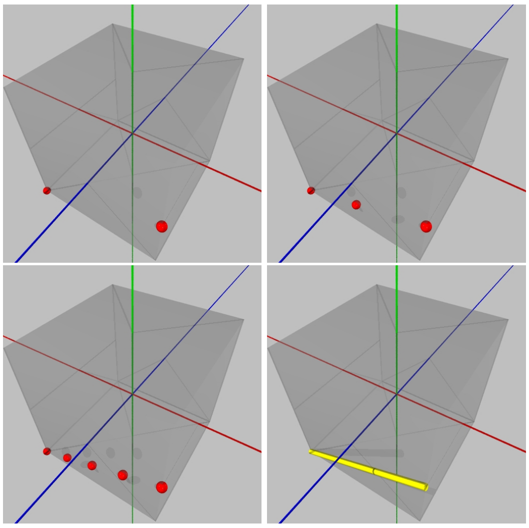

Figure 1: *two points lead to a segment (pL2)*

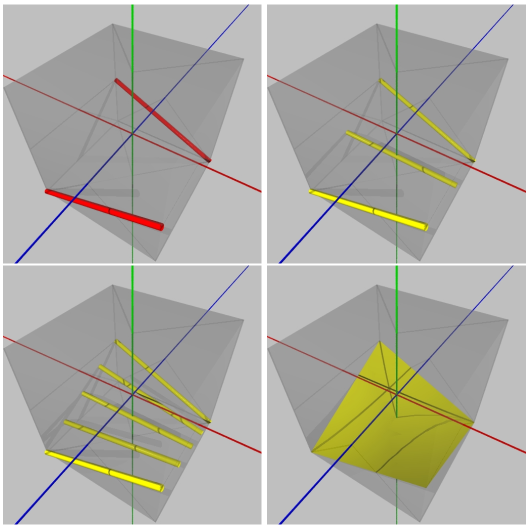

Figure 2: *two segments lead to a hyperbolic paraboloid (pS22)*

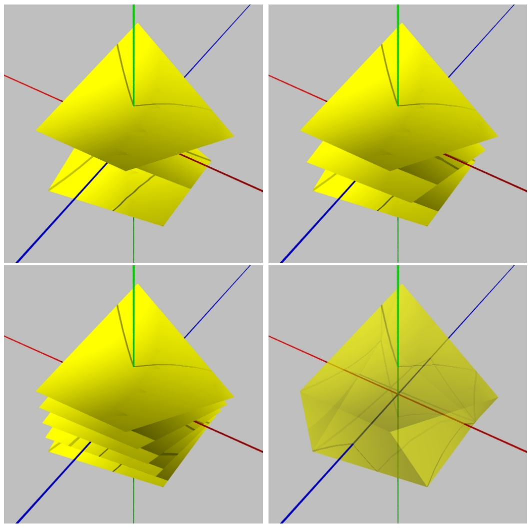

Figure 3: *two hyperbolic paraboloid lead to a twisted cube (pV222)*

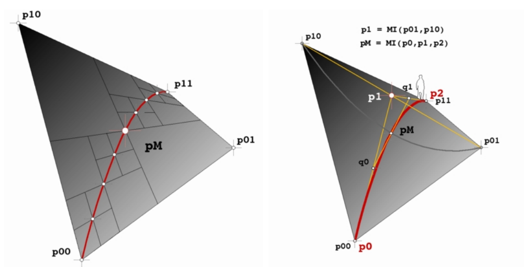

Figure 4: *the diagonal path on a pS22 is a parabola (pL3)*

a parabola (pL3) from the point of view of the embedding space in which is defined the pS22.

1.2. A ruled paraboloid (pS32) and its diagonal path (pL4)

In the same way that we have built a hyperbolic paraboloid (pS22) from 2 segments (pL2), we can now build a ruled paraboloid (pS32) from 2 parabolas (Figure 6, top left) or from 3 segments (Figure 6, top right) and draw a diagonal path . The diagonal path is a 4-controlled points curve (pL4), known as a cubic curve, whose control points are linked to the control points of the 2 parabolas in this way (Figure 6, bottom) :

$$p0 = p00,$$
$$p1 = (2*p01+p10)/3,$$
$$p2 = (p02+2*p11)/3,$$
$$p3 = p12.$$

and the middle point is expressed like this :

$$pM = (p0+3*p1+3*p2+p3)/8$$

The cubic (pL4) is the first true spatial curve (Figure 7, left) and can be viewed as the diagonal curve of the diagonal surface (pS32) of a twisted cube (pV222).

1.3. A pS33 and its diagonal path (pL5)

We will now build another surface from 3 parabolas (pS33), and draw a diagonal path (Figure 8). The diagonal path is a 5-controlled points curve (pL5), whose control points are

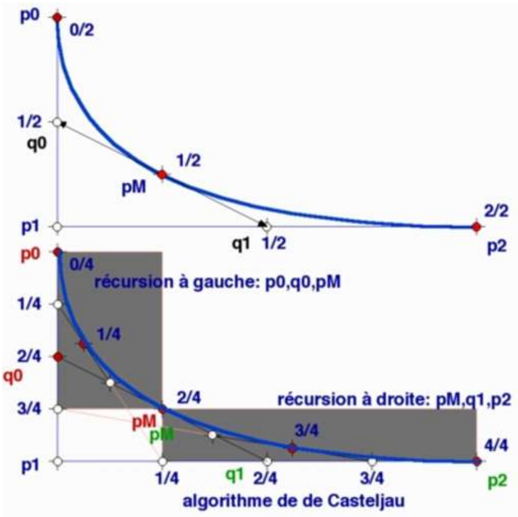

Figure 5: *the de Casteljau algorithm*

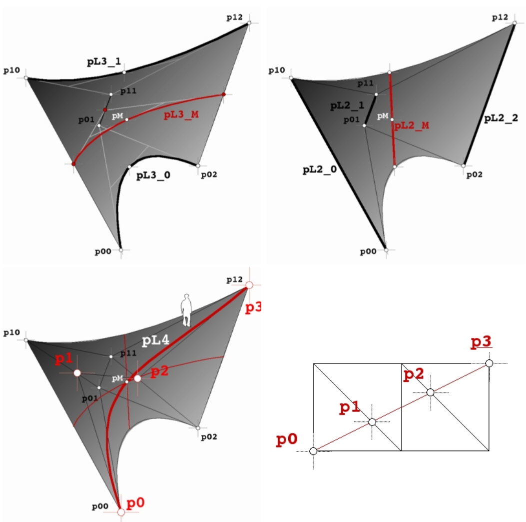

Figure 6: *a ruled paraboloid (pS32) from 2 pL3 or from 3 pL2, the diagonal path is a cubic curve (pL4), its 4 control points from the pS32*

linked to the 9 control points of the 3 parabolas in this way (Figure 9) :

$$p0 = p00,$$
$$p1 = (p01 + p10)/2,$$
$$p2 = (p02 + 4 * p11 + p20)/6,$$
$$p3 = (p12 + p21)/2,$$
$$p4 = p22.$$

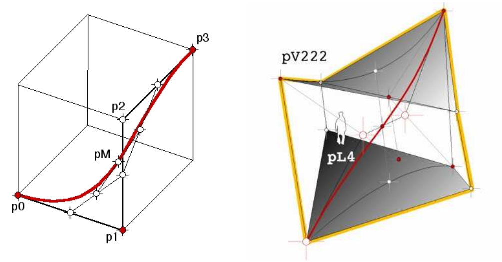

Figure 7: *the cubic shown in a cube, and as the second diagonal of a twisted cube.*

and the middle point is expressed like this :

$$pM = (p0 + 4 * p1 + 6 * p2 + 4 * p3 + p4)/16$$

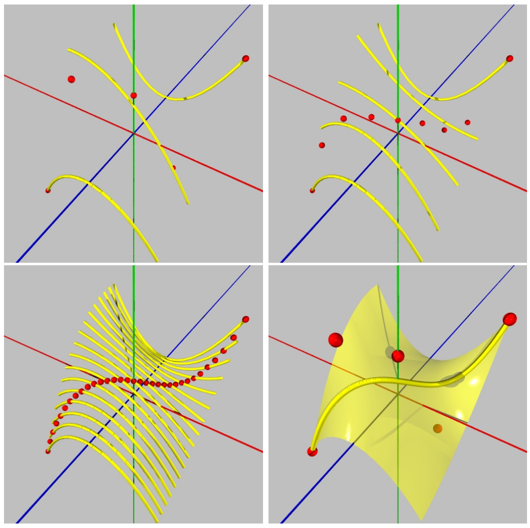

Figure 8: *three parabolas lead to the simplest double curved surface (pS33) and to a very useful curve (pL5)*

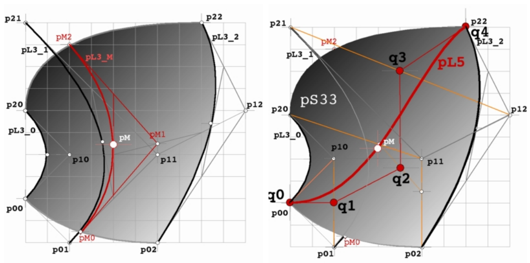

Figure 9: *building a pS33 and the 5 control points of its diagonal path*

1.4. The pForms

On the first family of recursive multilinear shapes, we have built new shapes (curves, surfaces, ...) and have enlighted the expression of the middle point of the diagonal paths of a pS22, a pS32, a pS33 :

$$pM = (p0 + p1)/2 \text{ (the middle of 2 points)}$$
$$pM = (p0 + 2 * p1 + p2)/4$$
$$pM = (p0 + 3 * p1 + 3 * p2 + p3)/8$$
$$pM = (p0 + 4 * p1 + 6 * p2 + 4 * p3 + p4)/16$$

which leads to the general expression of the middle of n forms (points, surfaces, volumes, ...) :

$$F_m = \sum_{i=0}^{n} C_{n-1}^{i} \times F^{i}/2^{n-1}$$
$$\text{with} : C_n^i = \frac{n!}{i! \times (n-i)!}$$

based on the Pascal's triangle coefficients. It is not surprising at all, all these forms are nothing but instances of well known Bezier shapes whose polynomial expressions use such coefficients ! But there are some good reasons to forget Bezier shapes, and to call them *pascalian forms* (or pForms to be short), pCurves, pSurfaces, pVolumes, pHyperVolumes. A first reason is the way they are built, grace to the strong link between the shapes and their diagonal paths, the shapes are built using the middle operator (product) to go up in dimension, or using the diagonal operator (divide), to go down in dimension. A second reason is that using recursive call of a the middle operator avoids boring analytic expressions used to define Bezier shapes : a Bezier volume is defined by a triple cartesian product of polynoms, a pVolume is the result of the middle operator applied recursively to a set of pSurfaces. But there is a more important reason to do this...

1.5. Immersions

The diagonal path links 2 opposites corners of a pSurface, but any 2 points in a pSurface can be connected in the same way : *stretching up or down the pSurface to fit with these 2 points brings back to the previous case*. The result is that we have got a way to define an unique curve connecting 2 points in a pSurface ; we call it an immersed segment (or iSegment or ipL2/pSurface) ; this pCurve becomes the fundamental element to build some geometry in the pSurfaces, and more generally in pForms of any dimensions. A first idea will be to define pPolylines, pTriangles, regular polygons and, when the number of sides grows to the infini, to define pCircles embedded in pSurfaces. Another better idea will be to define pCurves as we do in the space. For instance, with 3 points in a hyperbolic paraboloid (pS22), it is obvious to build an immersed parabola (ipL3/pS22), using nothing but basic recursives operations returning the middle point of immersed segments (Figure 10). More'over, it can be shown that in the most general case, *a pForm immersed in an other pForm is*

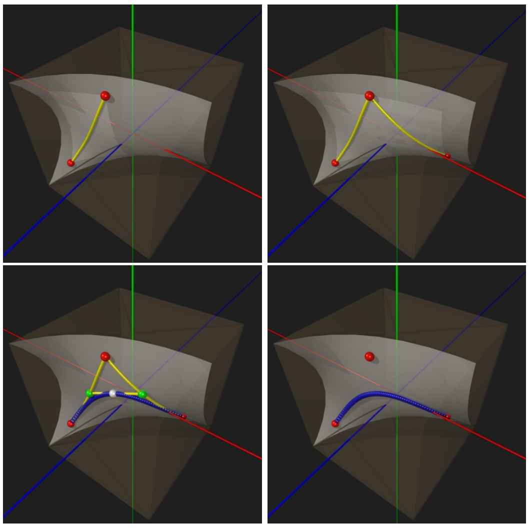

Figure 10: *three points in a pSurface lead to an immersed parabola (ipL3/pS22)*

a pForm of the initial space. This is a fundamental result leading to powerful properties. For instance, considering the special case of curves in surfaces, a pCurve controlled by p points embedded in a pSurface controlled by [m,n] points is still a pCurve of the space controlled by N points according to this formula :

$$N = (m + n - 2) * (p - 1) + 1$$

For instance, in a pS33, we find again that an immersed segment (ipL2) is a pCurve controlled by (3+3-2)*(2-1)+1 = 5 points (pL5) ; a parabola (pL3) will be a pCurve controlled by (3+3-2)*(3-1)+1 = 9 points (pL9) (Figure 11, left), and a cubic curve (ipL4) will be a pL13 (Figure 11, right). The

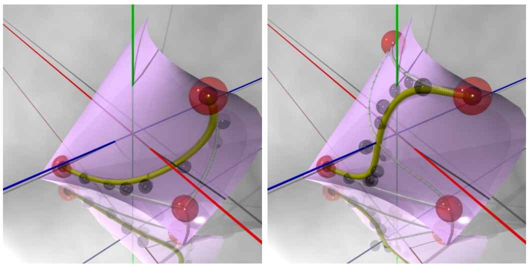

Figure 11: *ipL3/pS33 = pL9 ; ipL4/pS33 = pL13*

pForms cover a great family of forms whose (hidden) analytical expressions are complex cartesian products of polynomial expressions. Using the pForms approach, it becomes quite easy to draw some geometry in such shapes (Figure 12). We have now to remember that the useful *revolving surfaces* can't be expressed as polynomials expressions, and we

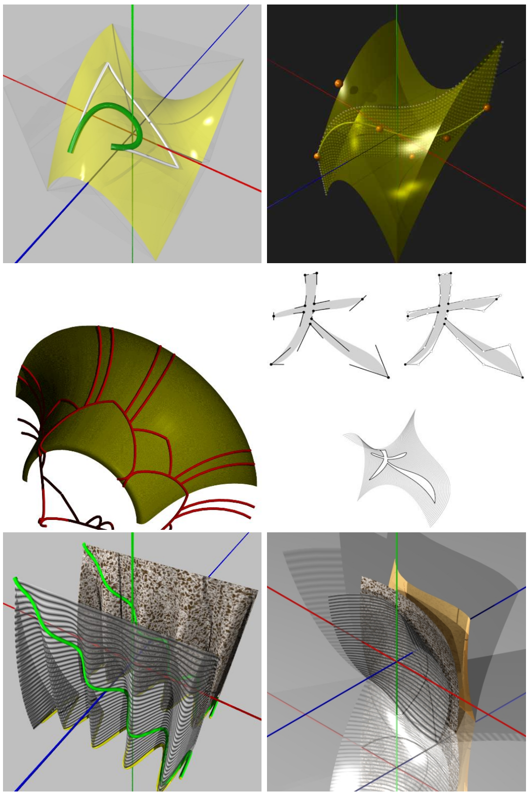

Figure 12: *some examples pf immersions and pForms*

have to do a little extra work to see them as pSurfaces, beginning with the circle.

2. The pForms, conics

The question is to see how a pCurve and a pSurface can be defined to represent respectively a circle and revolving surfaces, such as a torus. The parametric equation of a circle [cos(t), sin(t)] is irreducible to a vectorial polynom of finite degree, and it seems that no combination of points exists leading a pCurve to be the exact definition of a circle needed by CAD systems. The *Theory of Conics* shows that the intersection of a circular based cone and a plane parallel to a straight line on the cone is a parabola ; hence, with a special choice of three points of the R3 space, a parabola (pL3) will project on a plane (R2) as a perfect arc of a circle. According to this, *we will define pForms in the R4 space and get their projections in our R3 space*. Coming back to the circle, we have to go a little further. We have got a solution to a good quarter of the circle (Figure 13, top left), but it appears that the extension to the entire circle (360ř) uses an inappropriate infinite interval. Happily, beyond parabolas, other curves can be defined on a torus projecting as an arc of a circle, and a good choice is the 5 points controlled pCurve (pL5) we have already seen as a pS33' diagonal, (Figure 13, bottom left) ; this curve is projected as a true semi-circle and its extension to the entire circle is done in a reduced interval [-k,1+k] with k=sqrt(2)/2=0.707, producing a good repartition of the points on the curve. With such a well defined pCircle (belonging now to the family of pCurves), it becomes possible to build true and complete revolving pSurfaces, for instance a pTorus, in which it will be possible to define immersed pSegments and, why not, immersed pCircles which are nothing but pCurves of the space (Figures 14 and 15). Beyond these basic shapes, pForms can be used as compo-

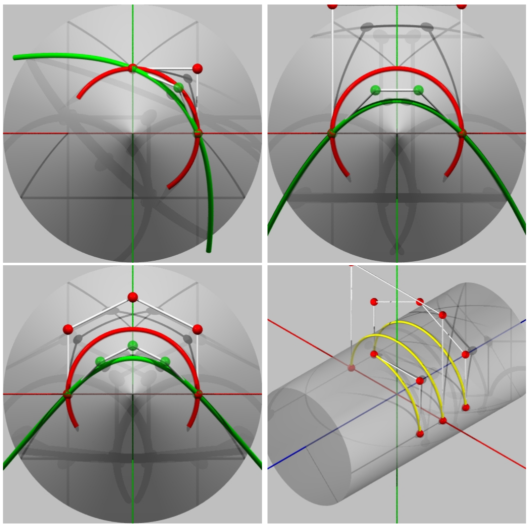

Figure 13: *pL3, pL4 and pL5 in a 4-dimensional space lead to true circles arcs*

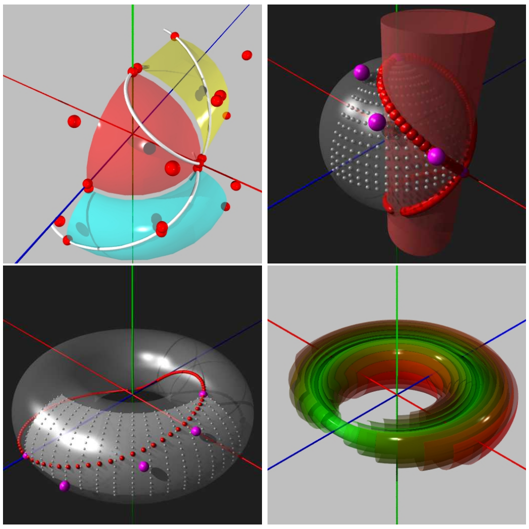

Figure 14: *revolving surfaces (pS33 in R4) and their diagonals (pL5)*

nents of useful linear combinations : concatenations lead to splines (uniform or non uniform B-splines), and to NURBS (we are working in R4 space) ; with special linear combinations it is possible to build Coons pSurfaces (pS1+pS2-pS3) ; pVolumes can be used to bend and to twist in a global way pCurves and pSurfaces. Because of the non metric definition of all the pForms, a complete operative geometry in curved spaces can be built upon them. The pForms can be used as this, and as basis in the approximation of complex mathematical or physical and natural shapes, giving an operative and intuitive tool allowing their analysis. It is what we are going to show for three cases in the real world.

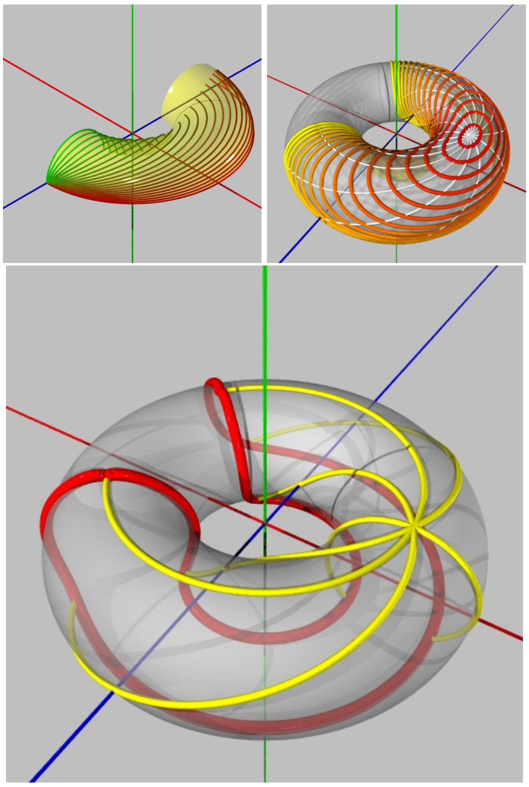

Figure 15: *immersed ina pTorus, pencil of pSegments, concentric pCircles and a red pCircle with some yellow rays*

3. Geodesical arcs on a swimming pool

The first case shows how a geodesic can be the solution of a very practical problem : *applying a long thin plank on a curved surface*. This problem came in the definition of a roof upon a public circular swimming pool in St Quentin in Yvelines (France), taken in charge by the engineer agency of Michael Flach specialized in wooden structures. The structural principle developed by this engineer is quite elegant and *a priori* simple : long thin planks (45mm/250mm) are alternaltively stacked to constitute arcs of a great inertia (8 planks giving an arc of section 360mm/250mm) perpendicular to the surface to be created, allowing an easy crossing of two families of these arcs without extra problems on knots, the whole becoming a hyperstatic mesh on which the thin plates of the roof can be layered (Figure 17). The *little residual problem* remaining was indeed to lay these long thin planks on a curved shape following curves perpendicular to the surface, and to give some help to the carpenter to do the job on the site. Immersed segments are (generally) not curves whose principal normal is perpendicular to the surface. Geodesic lines have this property, but their definition need to resolve a complex set of partial differential equations. A short insight will help to understand. A point M of

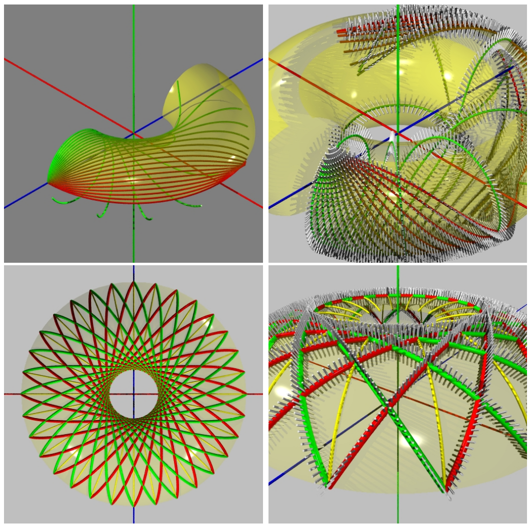

Figure 16: *pencils of geodesic lines on a torus, with and without normals ; study of the toroidal roof, geodesical arcs with and without normals*

a surface is defined by these expressions (input) :

$$M(u,v) = [x(u,v), y(u,v), z(u,v)] \text{ with : u,v in [0,1]}$$

known as parametric equations of the surface. This point M can alternatively be seen as belonging to a curve embedded in the surface, following this parametric expression to find (output) :

$$M(t) = [u(t), v(t)] \text{ with : t in [0,1]}$$

The first derivative (speed) of M(t) related to the variable t can be expressed with the two partial derivatives (tangent vectors) at the point M(t) of the surface :

$$dM/dt = \partial M/\partial_u * du/dt + \partial M/\partial_v * dv/dt$$

or shortly :

$$M' = \partial_u M.u' + \partial_v M.v'$$

We compute the second derivative (acceleration) :

$$\begin{aligned} M" &= dM'/dt \\ &= d(\partial_u M'.u' + \partial_v M'.v')/dt \\ &= \partial_u M * u" + \partial_v M * v" \\ &\quad + \partial^2_{uu} M * u'^2 + 2 * \partial^2_{uv} M.u'.v' + \partial^2_{vv} M.v'^2 \end{aligned}$$

and we express the orthogonality of the tangent vectors and the acceleration vector by zeroing their dot product :

$$M" * \partial_u M = 0,$$
$$M" * \partial_v M = 0$$

This is the partial differential system whose solution is a geodesic line (Figure 16). Unfortunaltely, there is no known

Figure 17: *realization, outside and inside*

general solution to this problem and it is necessary to proceed in an iterative way. The algorithm used to find the geodesics leads to a set of points, which is not a well defined curve as can be the pCurves : the solution is nothing but a terminal object, which can't be used as a tool to build new geometrical shapes, as it can be done with immersed segments or other pForms. We go straight away on a surface, but blindly and in a quite chaotic way (as it can be seen on the Figure 16, top). It could be interesting to see how immersed segments can be a good basis to simplify the analyze of geodesics ; the question is opened.

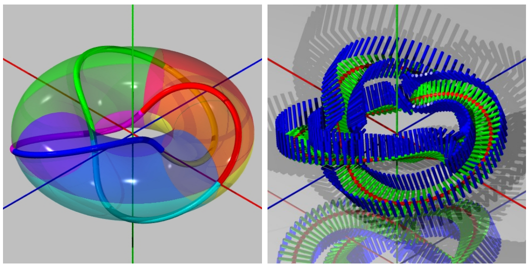

Figure 18: *concatenation of six iSegments, the study of the steps and the railings of the stairs*

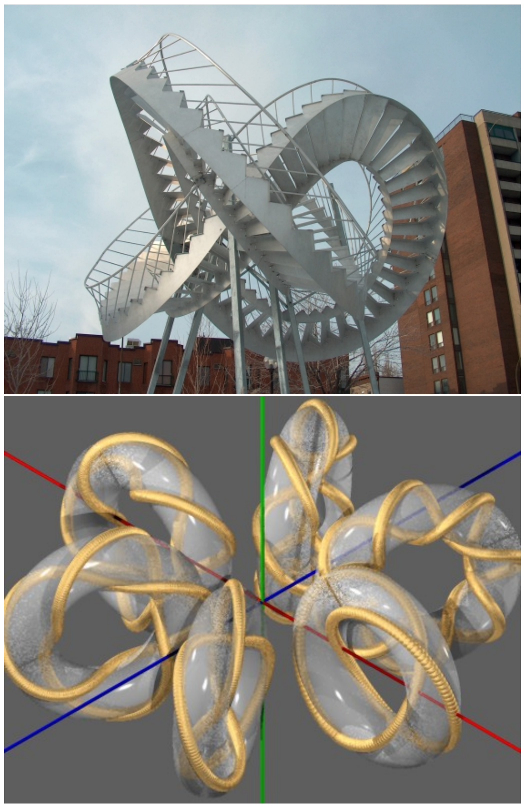

Figure 19: *strange MC Escher-like stairs in Montreal, and knots from 2 to 7*

4. A threefoil knot in Montreal

The second case we are going to analyse shows how an apparently complex spatial curve (the Threefoil Knot Curve) can be simplified by using simple immersed segments in a pTorus. Staircases in Montreal are known to be quite surprising, climbing up on the front of little buildings of two or three stacked apartments, twisting themselves to deserve every floor without obstruct too much the windows looking on the street, reminding us the impossible stairs imagined by M.C. Escher. The architect Guillaume Labelle (website: http://ingallian.design.uqam.ca/goo/), won a concourse upon the idea to realize a stair along a curve known as the Three-foil Knot Curve. Even if the analytical expression of this curve remains simple in Euclidean space :

$$x(t) = (2+cos(3t/2)).cos(t)$$
$$y(t) = (2+cos(3t/2)).sin(t)$$
$$z(t) = sin(3t/2)$$

the definition of the tangent axes (Serret-Frenet tri-axis TNB) needed to design the steps and the railings uses some expressions which become complex and out a normal human scope (just good enough for computers...). Viewed as a concatenation of six immersed segments (ipL2) in a pTorus (Figure 18, left), this curve becomes easier to be analysed, manipulated and displayed with the distribution of the local Serret-Frenet axis (Figure 18, right). This approach appeared to be useful for the final tuning up of the steps and the railings of this stair (Figure 19, top). And beyond, it may be of some utility to wonder about the topology of the Universe, (Figure 19, bottom) which appears to be toroidal to some modern researchers in cosmology, and in which an infinite straight line (followed by a light beam) could be a more or less complex knot. Is our Universe a space-time donut or a bretzel cake... ?

5. Funicular shapes in Sagrada Familia

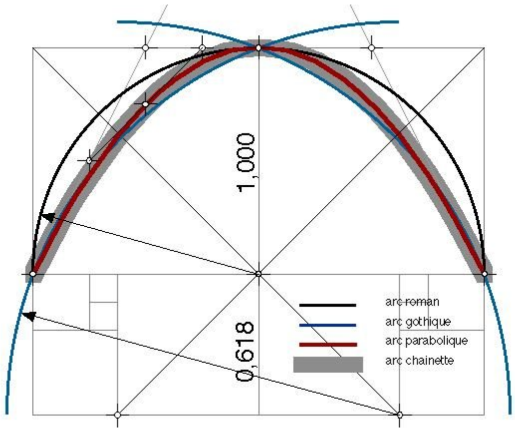

Figure 20: *evolution from the circle arc to the string arc*

This is the equation of the string curve and its polynomial development to the 6th order :

$$\begin{aligned} y &= cosh(x) \\ &= (e^{+x}+e^{-x})/2 \\ &= 1+x^2/2!+x^4/4!+x^6/6!+\ldots \end{aligned}$$

It is well known that the curve followed by a chain tensed under its own weight between two points in space is a funicular curve, a straight line in the gravitational field, and that reversing it upside/downside leads to structural arcs optimizing compression stresses. The problem encountered in implementing such a geodesic curve in real world comes from its transcendantal equation, and it is quite difficult to use it in real life (say a building site).

Figure 21: *from natural shapes to controlled shapes*

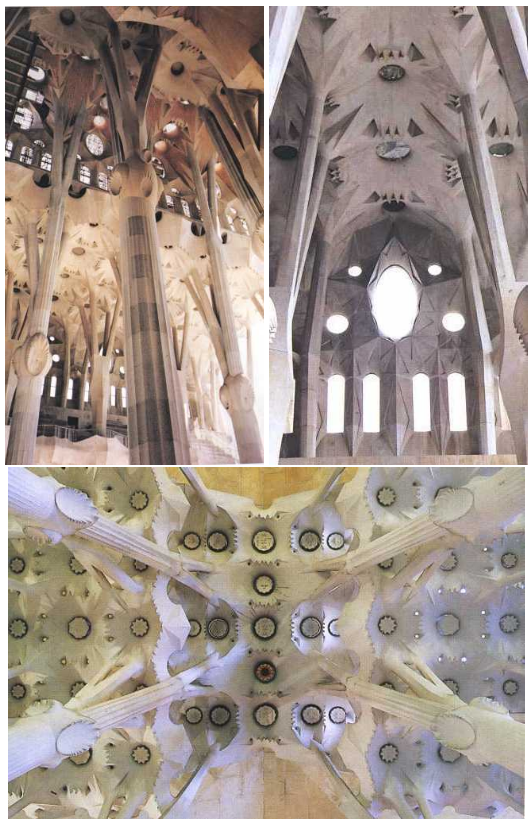

Figure 22: *complex combinations of ruled shapes*

Ancient Roman architects used plain circle arcs as a rough approximation easy to master in situ, Gothic architects went

Figure 23: *Gaudi's Sagrada Familia Temple in Barcelona*

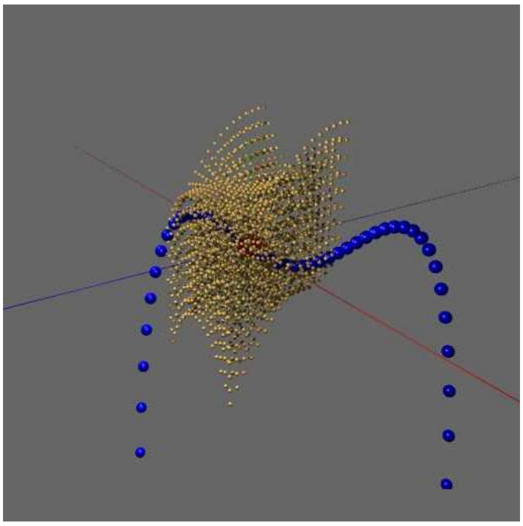

Figure 24: *a straight line in a curved space*

a little closer with a double circle arc, and the Catalan architect Antonio Gaudi definitively abandoned circles for the parabola, a very good approximation of the chain curve as it can be seen on the polynomial development shown above (Figure 20). Gaudi used to say : *God gives me the plan*, *Nature gives me the elevation*, *This tree in front of my workshop is my teacher*. For the conception of the Sagrada Familia Temple in Barcelona, he first built physical models using suspended funicular meshes supporting distributed weights, then solidified by plaster, inversed them to get the elevation, and began to enter in the design of the columns and the vaults with the image of a forest. At first sight, it could be thought that Gaudi's architecture is organic, made of true shapes coming from nature. But in fact, Gaudi followed but did not mimick the natural shapes (Figure 21) and in the ten last years of his life, he defined a powerful geometric vocabulary made of complex combinations of hyperbolic paraboloids and paraboloids of revolution and parabolic curves (Figure 22).

Contrary to his previous organic known language, such a vocabulay can be shared and it is required when the working process has to be done collectively. More'over, on these ruled surfaces (they are pSurfaces), beyond straight lines which make building possible, it becomes possible to draw immersed segments, immersed parabolas, immersed circles, leading to a complete geometry in curved space. The illustrations of the Sagrada Familia Temple come from [BON03].

6. Conclusion

Geodesic lines followed by thin planks on curved surfaces, complex mathematical curves as threefoil knots, funicular curves optimizing efforts, are certainly the true solutions but they come with a lot of mathematical complexities, making them difficult to be understood, to be controlled and to be used in the building of new geometrical objects. Going straight away in general curved spaces is not so easy ! Reducing curved shapes to pForms makes possible the definition of a a simple geometry in curved shapes, beginning with immersed segments, and provides valuable alternative geometric tools easier to define, manipulate and combine. These tools may help to fly straight away in curved shapes, to rediscover from another point of view the figures of the classical geometry, to help in the comprehension of modern theories in geometry, so often inaccessibles to the non mathematician and may be, to open a window on new ones (Figure 24)...

More can be seen at : http://marty.alain.free.fr/ (/recherche/formes pascaliennes/)

REFERENCES :

[CAS59] DE CASTELJAU, PAUL (1959), Courbes et surfaces a poles, Citroen, Paris

[FAR92] FARIN G, edition Masson (1992) Courbes et Surfaces pour la CGAO

[LES96] LES PIEGL WAYNE TILLER, edition Springer (1996) The NURBS Book

[FAR99] FARIN G, edition AK Peters, Ltd (1999) NURBS, from Projective Geometry to Practical Use

[POV99] POVRAY, (1999) POV-Help, POV-Team, at : http://www.povray.org

[BON03] JORDI BONET, Temple Sagrada Familia, Editorial Escudo de Oro 2003

[MAR04] MARTY A, edition de l'Esperou (2004) Formes Pascaliennes